NON-ANALYTIC SPINOR REPRESENTATION OF 4-DIMENSIONAL LORENTZ TRANSFORMATION

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ABSTRACT. This gives a representation of a 4-dimensional Lorentz transformation by utilizing the complex transformation coefficients of a unimodular unitary spinor transformation defined in a non-analytic complex space S_2. The treatment is similar to that followed in an earlier paper where the spinor representation of 5-dimensional Lorentz transformation was considered.

INTRODUCTION

In a complex two-dimensional non-analytic space S_2 a special transformation scheme called 'unimodular unitary non-analytic spinor transformation' has been framed. A mixed spinor $K^{\mu}{}_{\dot{\nu}}$ undergoing this transformation is then shown to be associated with a real vector K_i undergoing a 4-dimensional Lorentz transformation yielding the connecting relations between the respective transformation coefficients. In a recent paper (Ghosh 1965) the 'analytic spinor representation of 4-dimensional Lorentz transformation has been considered and some general properties of such spinors studied.

1. In an earlier paper (Ghosh, 1962) the general non-analytic spinor transformation scheme in S_2 has been formulated. Referring to § 2 of the next paper (Ghosh, 1964) one can define a unitary non-analytic spinor transformation in S_2 by postulating the invariance of an elementary spinor $\eta_{\mu\nu}$ with components

$$\eta_{1\dot{1}} = \eta_{2\dot{2}} = i, \quad \eta_{\dot{1}\dot{2}} = \eta_{2\dot{1}} = 0, \quad \eta_{\alpha\beta} = 0 \qquad \dots \quad (1.1)$$

$$\eta_{\dot{\alpha}\dot{\beta}} = \text{conj}\, \eta_{\alpha\dot{\beta}}, \quad \eta_{\dot{\alpha}\dot{\beta}} = \text{conj}\, \eta_{\alpha\beta} = 0$$

undergoing the transformation given by (Ghosh, 1962)

$$\eta'_{\mu\nu} = \eta_{\alpha\beta}(\xi'_{\mu}X^{\alpha})(\xi'_{\nu}X^{\beta})(\alpha, \beta = 1, 2, \dot{1}\dot{2}) \qquad \dots \quad (1.2)$$

For the invariance of η'_{μ} under unitary non-analytic spinor transformation we put $\eta'_{\mu\nu} = \eta_{\mu\nu}$ in the above and obtain the set of conditions expressed in the following matrix form :

$$\begin{vmatrix} \alpha & \beta & \lambda & \mu \\ \gamma & \delta & \nu & \sigma \\ \dot{\lambda} & \dot{\mu} & \dot{\alpha} & \dot{\beta} \\ \dot{\nu} & \dot{\sigma} & \dot{\gamma} & \dot{\delta} \end{vmatrix} \begin{vmatrix} \dot{\alpha} & \dot{\gamma} & -\lambda & -\dot{\nu} \\ \dot{\beta} & \dot{\delta} & -\dot{\mu} & -\sigma \\ -\dot{\lambda} & -\dot{\nu} & \alpha & \dot{\gamma} \\ -\dot{\mu} & -\dot{\sigma} & \dot{\beta} & \dot{\delta} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \qquad (1.3)$$

In the above $\alpha, \beta, \gamma, \delta$ denote the transformation coefficients ξ'_1X^1, ξ'_1X^2, $\xi'X^1$ ξ'_2x^2 refpectively and λ, μ, ν, σ the transformotion coefficients $\xi'_1 X^{\dot{1}}$, $\xi'_1X^{\dot{2}}$, $\xi'_2X^{\dot{1}}$ $\xi'_2x^{\dot{2}}$ respectively. Interchanging the order of the matrices in (1.3) we get an alternative set of conditions for unitary non-analytic spinor transformation The associated contravariant spinor $\eta^{\mu\nu}$ will be defined by the nonvanishing components

$$\eta^{1\dot{1}} = \eta^{2\dot{2}} = -i, \ \eta^{\dot{1}1} = n^{\dot{2}2} = i$$

so that

$$\eta^{\mu\nu}\eta_{\mu\rho} = \eta^{\nu\mu}\eta_{\nu u} = \delta^{\nu}{}_{\rho}. \qquad \ldots (1.4)$$

Raising and lowering of spinor indices will be performed under the scheme

$$\eta^{\nu\mu}\psi_{\mu} = \psi^{\nu}, \quad \psi^{\mu}\eta_{\mu\nu} = \psi_{\nu}.$$

Thus

$$\psi^1 = -i\psi_{\dot{1}}, \psi^2 = -i\psi_{\dot{2}}, \ \psi_1 = -i\psi^{\dot{1}}, \ \psi_2 = -i\psi^{\dot{2}} \qquad \ldots (1.5)$$

It may be noted that the contravariant and the covariant transformation coefficients are connected by the relations

$$\begin{aligned} \xi'_{\mu}X^{\alpha} &= \xi_{\dot{\alpha}}X'^{\dot{\mu}}, \\ \xi'_{\mu}\dot{X}^{\alpha} &= -\xi_{\alpha}\dot{X}'^{\mu}. \end{aligned} \qquad \ldots (1.6)$$

To obtain the appropriate spinor representation let us impose further restrictions on the unitary spinor field by postulating the invariance of an antisymmetric elementary spinor $\gamma_{\mu\nu}$ defined by the non-vanishing components

$$\gamma_{12} = -\gamma_{21} = i, \quad \gamma_{\dot{1}\dot{2}} = -\gamma_{\dot{2}\dot{1}} = -i. \qquad \ldots (1.7)$$

This leads to the following set of conditions expressed in matrix form :

$$\begin{bmatrix} \alpha & \beta & \lambda & \mu \\ \gamma & \delta & \nu & \sigma \\ \dot{\lambda} & \dot{\mu} & \dot{\alpha} & \dot{\beta} \\ \nu & \dot{\sigma} & \dot{\gamma} & \delta \end{bmatrix} \begin{bmatrix} \delta & -\beta & -\dot{\sigma} & \dot{\mu} \\ -\gamma & \sigma & \dot{\nu} & -\dot{\lambda} \\ -\sigma & \mu & \dot{\delta} & -\dot{\beta} \\ \nu & -\lambda & -\gamma & \alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (1.8)$$

In view of (1.3) this amounts to the additional conditions

$$\alpha = \delta, \ \beta = -\dot{\gamma}, \quad \lambda = \dot{\sigma}, \quad \mu = -\dot{\nu}. \qquad \ldots (1.9)$$

We shall call the unitary transformation thus modified 'unimodular unitary non-analytic spinor transformation'. Obviously, this set of transformations will possess group property.

2. Consider now a mixed spinor $K_{\nu}{}^{\mu}$ satisfying the structural equation

$$K_{\nu}{}^{\mu}\eta_{\mu\rho}\eta^{\nu\sigma} = K_{\rho}{}^{\sigma}. \qquad \ldots (2.1)$$

) that

$$K_1{}^1 = K_{\dot{1}}{}^{\dot{1}}\ ,\quad K_2{}^2 = K_{\dot{2}}{}^{\dot{2}}$$

$$K_2{}^1 = K_{\dot{1}}{}^{\dot{2}}\ ,\ K_1{}^2 = K_{\dot{2}}{}^{\dot{1}}\ ,$$

$$K_{\dot{1}}{}^1 = 0,\quad K_{\dot{2}}{}^2 = 0, \qquad \ldots \quad (2\,2)$$

$$K_{\dot{2}}{}^1 = -K_{\dot{1}}{}^2\ ,\quad K_1{}^{\dot{2}} = -K_2{}^{\dot{1}}\ .$$

Applying the rule (1.5) of raising and lowering of spinor indices one can see that $K_\nu{}^\mu$ will have the above structure if it is taken in the bilinear form

$$K_\nu{}^\mu = \psi^\mu \chi_\nu - \chi^\mu \psi_\nu, \qquad \ldots \quad (2.3)$$

ψ_μ, χ_ν being arbitrary spinors of rank 1.

Let us now express the components of the mixed spinor $K_\nu{}^\mu$ in terms of four real quantities k_i in conformity with the structural relation (2.2) in the following way :

$$K_1{}^1 = k_3,\quad K_2{}^2 = -k_3,$$

$$K_2{}^1 = k_1+ik_2,\quad K_1{}^2 = k_1-ik_2, \qquad \ldots \quad (2.4)$$

$$K_{\dot{2}}^1 = ik_0,\quad k_{\dot{1}}^2 = -ik_0$$

Here the invariant $K_\mu{}^\mu$ is assumed to be zero.

Let $K_\nu{}^\mu$ undergo a unimodular unitary non-analytic spinor transformation in S_2 given by

$$K'_\nu{}^\mu = K_\beta{}^\alpha(\xi_\alpha X'^\mu)(\xi_\nu' X^\beta), \qquad \ldots \quad (2.5)$$

where the coefficients of transformation satisfy (1.3, 7, 9). It can be verified that after the transformation $K'_\nu{}^\mu$ will have the same structure as that of $K_\nu{}^\mu$ so that we can express $K'_\nu{}^\mu$ in the same way as (2.4),

Rewriting (2.5) in a more compact form as a transformation formula involving a set of 4 mutually independent components $K_1{}^1$, $K_2{}^1$, $K_1{}^2$, $K_{\dot{2}}^1$ of $K_\nu{}^\mu$ and then converting this into one involving k_i expressed in the form

$$k_i' = \rho_i{}^j k_j (i, j = 0, 1, 2, 3) \qquad \ldots \quad (2.6)$$

the real transformation coefficients $\rho_i{}^j$ are obtained as follows :

$$\rho_3{}^0 = i\dot{\alpha}\mu - i\alpha\dot{\mu} - i\dot{\beta}\lambda + i\beta\dot{\lambda}, \qquad \rho_0{}^0 = \dot{\alpha}\dot{\delta} - \dot{\beta}\dot{\gamma} + \dot{\lambda}\dot{\sigma} - \dot{\mu}\dot{\nu},$$

$$\rho_3{}^1 = \dot{\alpha}\beta - \dot{\mu}\lambda + \dot{\beta}\alpha - \dot{\lambda}\mu, \qquad \rho_0{}^1 = -i\dot{\alpha}\dot{\sigma} + i\dot{\mu}\dot{\gamma} - i\dot{\beta}\dot{\nu} + i\dot{\lambda}\dot{\delta},$$

$$\rho_3{}^2 = i\dot{\alpha}\beta - i\dot{\mu}\lambda - i\dot{\beta}\alpha + i\dot{\lambda}\mu, \qquad \rho_0{}^2 = \dot{\alpha}\dot{\sigma} - \dot{\mu}\dot{\gamma} - \dot{\beta}\dot{\nu} + \dot{\lambda}\dot{\delta},$$

$$\rho_3{}^3 = \alpha\dot{\alpha} - \beta\dot{\beta} - \lambda\dot{\lambda} + \mu\dot{\mu}, \qquad \rho_0{}^3 = -i\dot{\alpha}\dot{\nu} + i\dot{\beta}\dot{\sigma} + i\dot{\gamma}\dot{\lambda} - i\dot{\mu}\dot{\delta},$$

$$\rho_1{}^0 = \tfrac{1}{2}(i\dot{\alpha}\sigma - i\alpha\dot{\sigma} - i\dot{\beta}\nu + i\beta\dot{\nu} + i\dot{\lambda}\delta - i\lambda\dot{\delta} - i\dot{\mu}\gamma + i\mu\dot{\gamma}),$$

$$\rho_2{}^0 = \tfrac{1}{2}(\dot{\alpha}\sigma + \alpha\dot{\sigma} - \dot{\beta}\nu - \beta\nu + \dot{\lambda}\delta + \lambda\dot{\delta} - \dot{\mu}\gamma - \mu\dot{\gamma}),$$

$$\rho_1{}^1 = \tfrac{1}{2}(\dot{\alpha}\delta + \alpha\dot{\delta} - \dot{\mu}\nu - \mu\dot{\nu} + \dot{\gamma}\beta + \gamma\dot{\beta} - \dot{\sigma}\lambda - \sigma\dot{\lambda}),$$

$$\rho_2{}^1 = \tfrac{1}{2}(-i\dot{\alpha}\delta + i\alpha\dot{\delta} + i\mu\nu - i\mu\dot{\nu} + i\dot{\gamma}\beta - i\gamma\dot{\beta} - i\dot{\sigma}\lambda + i\sigma\dot{\lambda}),$$

$$\rho_1{}^2 = \tfrac{1}{2}(i\dot{\alpha}\delta - i\alpha\dot{\delta} - i\dot{\mu}\nu + i\mu\dot{\nu} + i\dot{\gamma}\beta - i\gamma\dot{\beta} - i\dot{\sigma}\lambda + i\sigma\dot{\lambda}),$$

$$\rho_2{}^2 = \tfrac{1}{2}(\dot{\alpha}\delta + \alpha\dot{\delta} - \dot{\mu}\nu - \mu\dot{\nu} - \dot{\gamma}\beta - \gamma\dot{\beta} + \dot{\sigma}\lambda + \sigma\dot{\lambda}),$$

$$\rho_1{}^3 = \tfrac{1}{2}(\dot{\alpha}\gamma + \alpha\dot{\gamma} - \dot{\beta}\delta - \beta\dot{\delta} - \dot{\lambda}\nu - \lambda\dot{\nu} + \dot{\mu}\sigma + \mu\dot{\sigma}),$$

$$\rho_2{}^3 = \tfrac{1}{2}(-i\dot{\alpha}\gamma + i\alpha\dot{\gamma} + i\dot{\beta}\delta - i\beta\dot{\delta} + i\dot{\lambda}\nu - i\lambda\dot{\nu} - i\dot{\mu}\sigma + i\mu\dot{\sigma}),$$

Observing now that the invariant corresponding to $K_\nu{}^\mu$, namely,

$$K_\nu{}^\mu K_\mu{}^\nu = 4(k_1{}^2 + k_2{}^2 + k_3{}^2 - k_0{}^2) \tag{2.7}$$

we remark when $K_\nu{}^\mu$ undergoes a unimodular unitary non-analytic spinor transformation the induced real transformation on k_i is a 4-dimensional Lorentz transformation. It may be noted that the conditions (1.3, 9) imposed on the transformation coefficients in the representation (2.7) is equivalent to the exact number of conditions required.

On going over to the auxiliary real 4-space R_4 one can frame (Ghosh, 1964) the particular transformation which corresponds to the–unimodular unitary non-analytic spinor transformation in S_2. The mixed tensor which is obtained as 'analogue' of $K_\nu{}^\mu$ while undergoing this tensor transformation in R_4 will then induce a 4-dimensional Lorentz transformation to a vector, yielding the connection formulae of the representation expressed in real terms.

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